Errors-in-Variables Identification in Dynamic Networks -
Consistency Results for an Instrumental Variable Approach

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Abstract

In this paper we consider the identification of a linear module that is embedded in a dynamic network using noisy measurements of the internal variables of the network. This is an extension of the errors-in-variables (EIV) identification framework to the case of dynamic networks. The consequence of measuring the variables with sensor noise is that the prediction error identification methods no longer result in consistent estimates. The method proposed in this paper is based on a combination of the instrumental variable philosophy and closed-loop prediction error identification methods. We consider a flexible choice of which internal variables need to be measured in order to identify the module of interest. This allows for a flexible sensor placement scheme. We also present a method that can be used to validate the identified model.

1 Introduction

Systems in engineering are becoming more complex and interconnected. Consider for example, power systems, telecommunication systems, flexible mechanical structures and distributed control systems. Models of these networks are important either for physical parameter estimation, prediction, simulation or controller design. Fortunately sensors are becoming more ubiquitous and cheaper with the result that data can be collected from many variables in an interconnected dynamic network. Of course, every sensor measures a variable with noise. This measurement error is referred to as sensor noise (which is different from process noise which is caused by unknown disturbances exciting the network). In this paper methods are presented that use this noisy data to infer a model of (part of) a dynamic network.

In Van den Hof et al. (2013); Dankers et al. (2014b) methods have been presented that can be used to identify a particular module embedded in a dynamic network in the case that the interconnection structure of the network is known, and noise-free measurements of the internal variables are available (there is no sensor noise). In Materassi & Salapaka (2012); Yuan et al. (2011); Sandaji et al. (2012) methods have been presented that can be used in the case that the interconnection structure of the dynamic network is unknown. The main assumptions in these papers is that all nodes in the network are driven by independent process noise sources and sensor noise free measurements of all internal variables are available. In Massioni & Verhaegen (2008); Haber & Verhaegen (2012); Ali et al. (2011) conditions are presented for identifying modules in spatially distributed systems. Common assumptions in this literature is that each subsystem is identical (Massioni & Verhaegen, 2008; Ali et al., 2011), and/or that there is a known external excitation signal present at each node (Massioni & Verhaegen, 2008; Haber & Verhaegen, 2012; Ali et al., 2011), and that there is no process noise in the networks, only sensor noise.

In this paper we consider a very general framework where there may or may not be known external excitation present, there is both (correlated) process noise and (correlated) sensor noise present in the data collected from the network, the modules making up the network are not identical, and not all internal variables of the network are measureable. We make the assumption that the interconnection structure of the network is known. Including the possibility of sensor noise (in addition to process noise) is not as trivial as it may seem. In many identifi-
cation methods, the inputs are assumed to be measured without noise (Ljung, 1999). Moreover, if there is sensor noise on the inputs, the methods do not lead to consistent estimates.

Specifically, we consider the following question: under what conditions is it possible to consistently identify a particular module embedded in a dynamic network when only noisy measurements of a subset of the internal variables of the network are available? This is an extension of the so-called Errors-in-Variables framework to the case of dynamic networks.

The open loop EIV problem has been extensively studied (see the survey papers Söderström (2007, 2012)). The main conclusion in these papers is that either prior knowledge about the system or a controlled experimental setup is required to ensure consistent estimates. Examples of controlled experimental setups are presented in Söderström & Hong (2005) and Schoukens et al. (1997), where it is shown that using periodic excitation or repeated experiments it is possible to consistently estimate the plant in an open loop EIV setting (without any additional prior knowledge). The method proposed in Söderström & Hong (2005) to deal with the sensor noise is based on an Instrumental Variable (IV) method.

In this paper we also consider an IV based approach, and in a roughly similar vein to Söderström & Hong (2005), instead of using repeated experiments, we show that additional (noisy) measurements generated by the dynamic network can be used to deal with the sensor noise. It is the unique possibility of measuring extra variables in the network that is crucial for dealing with the sensor noise. The extra measurement(s) are used as instrumental variables. We present a method so that any extra measurements that are somehow correlated to the output of the module of interest can be used as instrumental variables, irrespective of their location in the network. However, due to the (possible) presence of loops the process noise must also be specifically dealt with.

In the closed-loop identification literature, it is well known that the problem with process noise in the presence of loops in the interconnection structure of the data generating system is that the predictor inputs are correlated to the process noise affecting the output Van den Hof (1998); Forssell & Ljung (1999). One closed-loop identification method is the Basic Closed-Loop Instrumental Variable (BCLIV) Method (Gilson & van den Hof, 1998; Forssell & Ljung, 1999). The main idea behind this method is that if the process noise in the loop can be exactly modelled, then it is possible to consistently identify the module of interest. However, the method does not lead to consistent estimates in the presence of sensor noise on the inputs.

The second method proposed in this paper is based on a combination of the Instrumental Variable method and the Direct closed-loop Prediction Error method. The problems caused by the process noise are dealt with using the reasoning of the direct closed-loop method, and the problems caused by the sensor noise are dealt with using an instrumental variable reasoning. In this method all measured variables that are not used to construct the predictor are candidate instrumental variables. The method can be cast as a generalization of the IV method using a one-step-ahead predictor model with a Box-Jenkins model structure. This method can also be cast as a generalization of the first method proposed in this paper. This paper is based, in part, on the preliminary results of Dankers et al. (2014a).

In Section 2 background material on dynamic networks, prediction error identification and instrumental variable methods is presented. The first method (a straightforward extension of the BCLIV method) is presented in Section 3, the second one (a combination of the IV and Direct Method) is presented in Section 4. In Section 5 the conditions on the locations in the network of the variables that must be measured are further relaxed. In Section 6 a practical implementation of the second method is proposed. In Section 7 a method is presented to validate the obtained model.

2 Background

2.1 Dynamic Networks

The framework considered in this paper is based on Van den Hof et al. (2013). A dynamic network is built up of \( L \) elements, related to \( L \) scalar internal variables \( w_j \), \( j = 1, \ldots, L \). Each internal variable is defined by:

\[
  w_j(t) = \sum_{k} G^0_{jk}(q) w_k(t) + r_j(t) + v_j(t) \tag{1}
\]

where \( G^0_{jk} \), \( k \in \mathcal{N}_j \) is a proper rational transfer function, \( q^{-1} \) is the delay operator, i.e. \( q^{-1} w_j(t) = w_j(t-1) \) and,

- \( \mathcal{N}_j \) is the set of indices of internal variables with direct causal connections to \( w_j \), i.e. \( k \in \mathcal{N}_j \) iff \( G^0_{jk} \neq 0 \);
• $v_j$ is **process noise**, that is modeled as a realization of a stationary stochastic process with rational spectral density: $v_j = H_j^0(q) e_j$ where $e_j$ is a white noise process, and $H_j^0$ is a monic, stable, minimum phase transfer function;
• $r_j$ is an **external variable** that is known to the user, and may be manipulated by the user.

It may be that the noise and/or external variables are not present at some nodes. The network is defined by:

$$
\begin{bmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_L \\
\end{bmatrix} =
\begin{bmatrix}
  0 & G_{12}^0 & \cdots & G_{1L}^0 \\
  G_{21}^0 & 0 & \cdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  G_{L1}^0 & G_{L2}^0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_L \\
\end{bmatrix} +
\begin{bmatrix}
  r_1 \\
  r_2 \\
  \vdots \\
  r_L \\
\end{bmatrix} +
\begin{bmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_L \\
\end{bmatrix}
$$

where $G_{jk}^0$ is non-zero if and only if $k \in \mathcal{N}_j$ for row $j$, and $v_k$ (or $r_k$) is zero if it is not present. Using an obvious notation this results in the matrix equation:

$$w = G^0 w + r + v. \quad (2)$$

Each internal variable is measured with some measurement or sensor error:

$$\tilde{w}_k(t) = w_k(t) + s_k(t), k = 1, \ldots, L$$

where $\tilde{w}_k$ denotes the measurement of $w_k$, and $s_k$ is the **sensor noise**, which is represented by a stationary stochastic process with rational spectral density ($s_k$ is not necessarily white noise).

There exists a **path** from $w_i$ to $w_j$ if there exist integers $n_1, \ldots, n_k$ such that $G_{jn_1}^0 G_{n_1 n_2}^0 \cdots G_{n_k j}^0$ is non-zero.

The following assumption holds throughout the paper.

**Assumption 1 General Conditions.**

(a) The network is well-posed in the sense that all principal minors of $\lim_{t \to \infty} (I - G^0(z))$ are non-zero.
(b) $(I - G^0)^{-1}$ is stable.
(c) All process noise variables $v_k$ are uncorrelated to all sensor noise variables $s_k$.

### 2.2 Prediction Error Identification

In this section some key results of the prediction error identification method are presented. See Ljung (1999) for a (much) more thorough treatment of the material. Let $w_j$ denote the variable which is to be predicted. The one-step-ahead predictor for $w_j$ is then (Ljung, 1999):

\[
\hat{w}_j(t | t - 1, \theta) = H_j^{-1}(q, \theta) \sum_{k \in \mathcal{D}_j} G_{jk}(q, \theta) \tilde{w}_k(t) + \left(1 - H_j^{-1}(q, \theta)\right) \hat{w}_j(t) \quad (3)
\]

where $H_j(q, \theta)$ is a (monic) noise model and $G_{jk}(q, \theta), k \in \mathcal{D}_j$ are module models. The **predictor inputs** are those variables that are used to predict $w_j$. The set $\mathcal{D}_j$ denotes the set of indices of the measurements that are chosen as predictor inputs, i.e. $\tilde{w}_k$ is a predictor input if $k \in \mathcal{D}_j$. The prediction error is:

\[
\varepsilon_j(t) = \hat{w}_j(t) - \hat{w}_j(t | t - 1, \theta) = H_j(q, \theta)^{-1} \left(\hat{w}_j(t) - \sum_{k \in \mathcal{D}_j} G_{jk}(q, \theta) \tilde{w}_k(t)\right). \quad (4)
\]

Usually the parameterized transfer functions $G_{jk}(\theta), k \in \mathcal{D}_j$, and $H_j(\theta)$ are estimated by minimizing the sum of squared (prediction) errors. Let $\hat{\theta}_N$ denote the estimated parameter vector based on $N$ data points. If $\hat{\theta}_N \to \theta^0$ as $N \to \infty$ with probability 1, then the obtained estimates are consistent.

The following notation will be used throughout the remainder of this paper. The auto and cross correlation of vectors of variables $x$ and $y$ are defined as

\[
R_x(\tau) := \mathbb{E} [x(t) x^T (t - \tau)],
R_{xy}(\tau) := \mathbb{E} [x(t) y^T (t - \tau)]
\]

respectively, where

\[
\mathbb{E}[\cdot] = \lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathbb{E}[\cdot]
\]

where $\mathbb{E}$ denotes the expected value operator. The power spectral density and cross power spectral density are

\[
\Phi_x(\omega) := \mathcal{F}[R_x(\tau)] \quad \text{and} \quad \Phi_{xy}(\omega) := \mathcal{F}[R_{xy}(\tau)]
\]

respectively, where $\mathcal{F} [\cdot]$ denotes the Fourier Transform.

### 2.3 Closed Loop Instrumental Variable Methods

It is well known that the main difficulty in closed-loop identification is that due to the feedback loop, the input is correlated to the process noise on the output (Ljung, 1999; Forssell & Ljung, 1999). The Instrumental Variable method can be seen as a generalization of the Least Squares Method (Söderström & Stoica, 1983). The
mechanism that ensures that the closed-loop IV methods result in consistent estimates is to correlate the input and output with a so-called instrumental variable. As long as the instrumental variable is uncorrelated to the process noise, consistent estimates are possible. A survey paper on closed-loop instrumental variable methods is Gilson & Van den Hof (2005). In the following text, the Basic Closed-Loop Instrumental Variable (BCLIV) method of Gilson & Van den Hof (2005) is presented.

A closed-loop data generating system is:

\[ w_2 = G_2^0 w_1 + v_2, \quad (6a) \]
\[ w_1 = G_1^0 w_2 + r_1. \quad (6b) \]

Suppose that there is no sensor noise. The objective is to obtain a consistent estimate of \( G_{21}^0 \). Consider an ARX model structure, i.e. the module transfer function \( G_{21}(\theta) \) is parameterized as (Ljung, 1999):

\[ G_{21}(\theta) = \frac{B_{21}(\theta)}{A_{21}(\theta)} = \frac{q^{-n_b} (b_0^{21} + \cdots + b_{n_b}^{21} q^{-n_b})}{1 + a_1^{21} q^{-1} + \cdots + a_{n_b}^{21} q^{-n_b}} \quad (7) \]

and the noise model is parameterized as \( H_{21}(\theta) = \frac{1}{A_{21}(\theta)} \).

Note that \( A_{21} \) is a polynomial of order \( n_a \) and \( B_{21} \) is a polynomial of order \( n_b \). The parameter vector \( \theta \) is

\[ \theta = [a_1^{21} \cdots a_{n_a}^{21} b_0^{21} \cdots b_{n_b}^{21}]^T. \quad (8) \]

From (4) the prediction error is

\[ \varepsilon_{2}(t, \theta) = A_{21}(q, \theta) w_2(t) - B_{21}(q, \theta) w_1(t) \quad (9) \]

which can be expressed as

\[ \varepsilon_{2}(t, \theta) = w_2(t) - \phi^T(t) \theta \quad (10) \]

where

\[ \phi^T(t) = \begin{bmatrix} -w_2(t-1) \cdots -w_2(t-n_a) w_1(t) \cdots w_1(t-n_b) \end{bmatrix}. \quad (11) \]

Let \( z \) denote the variable chosen as the instrumental variable. In the BCLIV method, \( r_1 \) is chosen to be the instrumental variable. The mechanism that forms the foundation of the BCLIV method is presented in the following proposition.

**Proposition 1 (BCLIV)** Consider a closed-loop system (6) that satisfies Assumption 1. Consider the prediction error (10). Let the instrumental variable \( z = r_1 \). The equivalence relation

\[ \left\{ \mathbb{E}[\varepsilon(t, \theta) z(t-\tau)] = 0, \text{ for } \tau = 0, \ldots, n \right\} \quad \iff \quad \left\{ G_{21}(q, \theta) = G_{21}^0(q) \right\}. \]

holds for any finite \( n \geq n_a + n_b + 1 \) if the following conditions are satisfied:

(a) \( \mathbb{E}[\phi(t) \cdot [z(t) \cdots z(t-n_a-n_b)]] \) is nonsingular,
(b) \( \mathbb{E} [\varepsilon_{2}(t, \theta) z(t-\tau)] = 0, \forall \tau \geq 0. \)
(c) The parameterization is chosen flexible enough, i.e. there exists a \( \theta \) such that \( G_{21}(q, \theta) = G_{21}^0(q) \). \( \Box \)

For a proof see Gilson & Van den Hof (2005). The main point is that the cross correlation between the instrumental variable and the prediction error is zero if and only if \( G_{21}(q, \theta) = G_{21}^0(q) \) (as long as the conditions hold). This implies that if a parameter vector \( \theta \) can be found that satisfies the set of equations \( R_{2z}(\tau), \tau = 0, 1, \ldots, n \) then \( \theta \) in fact characterizes a consistent estimate of \( G_{21}^0 \). This leads to the following algorithm.

**Algorithm 1**

1. Choose \( r_1 \) as the instrumental variable. Let \( z = r_1 \).
2. Choose an ARX model structure and construct the prediction error (10).
3. Find a solution to the set of equations

\[ \frac{1}{N} \sum_{t=0}^{N-1} \varepsilon_{2}(t, \theta) z(t-\tau) = 0, \tau = 0, \ldots, n_a + n_b \quad (12) \]

From Proposition 1 it follows that Algorithm 1 results in a consistent estimate of \( G_{21}^0 \). Note that in Step 3 of Algorithm 1 a solution to an approximation of \( R_{2z}(\tau) \) is obtained. The bigger \( N \) is, the better the approximation. Secondly, the solution to the set of equations (12) can be found by linear regression. This follows from (10) where it appears that (for the ARX model structure) the prediction error is linear in \( \theta \).

### 2.4 Persistently Exciting Data

Condition (a) of Proposition 1 has a very nice interpretation in terms of persistence of excitation of the data. In any identification method the (input) data must be persistently exciting of sufficiently high order to ensure a unique estimate (Ljung, 1999). In this section, first we provide the definition of persistence of excitation, and then it is shown that Condition (a) holds if the external variable \( r_1 \) is persistently exciting of sufficiently high order. This provides insight into which mechanisms ensure Condition (a) holds. The formalization presented in this section is used in the extension to the dynamic network case where, due to the presence of many different variables and complex interconnection structures the insight offered by the concept of persistence of excitation becomes increasingly useful. We employ the setting of Söderström & Stoica (1983, 1989).

Consider first the classical definition of persistence of excitation Söderström & Stoica (1989); Ljung (1999).
Definition 2 The vector of (internal and/or external) variables $u$ is persistently exciting of order $n$ if

$$
\hat{R}_u = \begin{bmatrix}
R_u(0) & \cdots & R_u(n-1) \\
\vdots & & \vdots \\
R_u(n-1) & \cdots & R_u(0)
\end{bmatrix}
$$

is positive definite, where $R_u(t)$ is the auto-correlation of the vector $u$ as defined in (5).

The concept of persistence of excitation imposes a constraint that the data must satisfy. For the BCLIV method presented in Section 2.3 the external variable $r_1$ must be persistently exciting of sufficiently high order to ensure that the matrix in Condition (a) of Proposition 1 is full rank. This is formalized by the following Lemma from Söderström & Stoica (1983, 1989).

Lemma 3 Consider a closed-loop system (6) that satisfies Assumption 1. Consider Algorithm 1. Consider the matrix

$$
R = \mathbb{E} \left[ \phi(t) \left[ z(t) \ z(t-1) \ \cdots \ z(t-n_z) \right] \right]
$$

where $z$ is the instrumental variable, and $\phi(t)$ is defined in (11). The matrix $R$ generically has rank $n_a + n_b + 1$ (i.e. full row rank) if the following conditions hold:

(a) $n_z \geq n_a + n_b + 1$.
(b) $r_1$ is (at least) persistently exciting of order $n_a + n_b + 1$.
(c) The model orders $n_a$ and $n_b$ are equal to the orders $n^0_a$ and $n^0_b$ that define $G^0_{21}$. \hfill $\Box$

For a proof see Lemma 4.1 and Theorem 4.1 in Söderström & Stoica (1983). The main point of Lemma 3 is to provide an interpretation of Condition (a) of Proposition 1. It provides the connection between persistence of excitation of the data and consistency of the estimate for IV methods. For a discussion on why the result only holds generically see Söderström & Stoica (1983, 1989).

Condition (a) of Lemma 3 ensures that the matrix $R$ has at least $n_a + n_b + 1$ columns. Clearly this is necessary in order for $R$ to have rank $n_a + n_b + 1$. Condition (c) of Lemma 3 ensures that there is a unique $\theta$ that represents $G^0_{21}$ (i.e. the model is identifiable in the sense that $G^0_{21}(\theta) = G^0_{21} \iff \theta = \theta_0$).

The persistence of excitation of a variable can also be characterized in the frequency domain. Consider the following two Propositions.

Proposition 4 Let $z$ be a scalar variable. The variable $z$ is persistently exciting of order $n$ if and only if the spectral density $\Phi_z$ is non-zero for at least $n$ distinct frequencies $\omega \in (-\pi, \pi)$. \hfill $\Box$

For a proof see Result A1.3 in Söderström & Stoica (1983).

Proposition 5 Let $z$ be a vector of variables. If the spectral density matrix is full rank for at least $n$ distinct frequencies $\omega \in (-\pi, \pi)$ then $z$ is persistently exciting of order $n$. \hfill $\Box$

For a proof see Result A1.1 in Söderström & Stoica (1983). Interestingly, in the case where $z$ is a vector the frequency domain condition implies that $z$ is persistently exciting, but not the other way around. Thus, there exist vectors of variables that are persistently exciting of order $n$ such that the spectral density is not full rank for at least $n$ distinct frequencies. For example

$$
z(t) = [\sin(\omega_1 t) \ \sin(\omega_2 t)]
$$

is persistently exciting of order 2, but $\Phi_z$ is rank deficient at all frequencies.

In the following section a method is presented for identification in networks that is a straightforward extension of the BCLIV method.

3 Extension of BCLIV Method to Dynamic Networks and sensor noise

Recall, the objective considered in this paper is to obtain an estimate of a particular module, $G^0_{ji}$, embedded in a dynamic network using noisy measurements of the internal variables. In this section a straightforward extension of the BCLIV method is presented. The extension focuses on three aspects:

- We generalize the method so that it is able to identify a particular module embedded in a dynamic network, not just a closed-loop data generating system.
- We consider the situation that all measured variables can be subject to sensor noise.
- Rather than the classical case where only external variables are considered as candidate instrumental variables, we consider both internal and external variables as candidate instrumental variables.

A main theme in this paper is that for a dynamic network, there are many different variables present that can serve as potential instrumental variables. For instance, one can choose between several external and internal variables. In this paper we consider any measured or known variable that is not $\tilde{w}_j$ or a predictor input as a potential instrumental variable. In other words, the set of candidate instrumental variables is $\tilde{w}_\ell, \ell \in \{1, \ldots, L\} \setminus \{D_j \cup \{j\}\}$ and $r_\ell, \ell \in R$. Let $\mathcal{X}_j$ and $\mathcal{I}_j$ denote the set of indices of external and internal variables respectively chosen as instrumental variables (i.e. $r_\ell$ is an instrumental variable iff $\ell \in \mathcal{X}_j$ and $\tilde{w}_\ell$ is...
an instrumental variable if \( \ell \in I_j \)). Since predictor inputs and \( \tilde{w}_j \) are not considered as allowable instrumental variables it must be that \( I_j \cap \{ D_j \cup \{ j \} \} = \emptyset \).

The variables that are selected as instrumental variables are placed in a vector of instrumental variables, denoted \( z \). Three methods for constructing \( z \) are suggested below.

- Choose one or more external variables, \( r_{\ell_1}, \ldots, r_{\ell_n} \), as instrumental variables, resulting in
  \[
  z(t) = [r_{\ell_1}(t) \cdots r_{\ell_n}(t)]^T
  \]
  and \( X_j = \{ \ell_1, \ldots, \ell_n \}, I_j = \emptyset \).
- Choose one or more measurements of internal variables, \( \tilde{w}_{\ell_{11}}, \ldots, \tilde{w}_{\ell_{dn}} \), as instrumental variables:
  \[
  z(t) = [\tilde{w}_{\ell_{11}}(t) \cdots \tilde{w}_{\ell_{dn}}(t)]^T
  \]
  and \( X_j = \emptyset, I_j = \{ \ell, 1, \ldots, \ell_n \} \).
- Choose sums of measured internal variables (or external variables, or a combination of both), \( \tilde{w}_{\ell_{11}} + \cdots + \tilde{w}_{\ell_{1m}} + \cdots + \tilde{w}_{\ell_{dn}} \). In this case
  \[
  z(t) = \left[ \sum_{m=1}^n \tilde{w}_{\ell_{1m}}(t) \cdots \sum_{m=0}^n \tilde{w}_{\ell_{dn}}(t) \right]^T
  \]
  and \( X_j = \emptyset, I_j = \{ \ell_{11}, \ldots, \ell_{1m}, \ldots, \ell_{dn} \} \).

Of course, combinations of these cases can also be used. Which method is used to choose the instrumental variables will depend on which variables are available/measured, and which choice ensures that \( z \) is persistently exciting of sufficiently high order.

In any instrumental variable method, it is essential that the instrumental variables and the predictor inputs are correlated. In the BLCIV method, the instrumental variable was chosen to be \( r_1 \), which is correlated to both \( w_1 \) and \( w_2 \). In the case of dynamic networks it is not automatically guaranteed that a candidate internal variable is correlated to (one or more of) the predictor inputs and/or \( w_j \). The following lemma presents graphical conditions to check whether two variables are correlated.

**Lemma 6** Consider a dynamic network as defined in Section 2.1 that satisfies Assumption 1. Let \( z_\ell \) be an internal or external variable. Then \( z_\ell \) and \( w_k \) are not correlated if the following three conditions hold:

(a) There is no path from \( z_\ell \) to \( w_k \).

(b) There is no path from \( w_k \) to \( z_\ell \).

(c) There is no variable \( w_p, p \notin D_j \cup I_j \cup \{ j \} \) such that there are paths from \( w_p \) to both \( z_\ell \) and to \( w_k \). \( \square \)

The proof can be found in Appendix A. This lemma can guide the user to choose appropriate instrumental variables that are correlated to the predictor inputs.

### 3.1 Generalization of BCLIV Method

As in Section 2.3 first a mechanism that forms the foundation of the method is presented. Then an algorithm is proposed to exploit this mechanism in order to obtain consistent estimates of a module embedded in a network.

The main modification that must be made to the BCLIV algorithm to be able to use it in dynamic networks is to move to a multiple input, single output (MISO) ARX model structure. For a MISO ARX model structure, the modules and noise model are parameterized as:

\[
G_{jk}(q, \theta) = \frac{B_{jk}(q, \theta)}{A_{j}(q, \theta)}, \quad H_{j}(q, \theta) = \frac{1}{A_{j}(q, \theta)},
\]

for all \( k \in D_j \), where

\[
B_{jk}(q, \theta) = q^{-n_k} (b_{0_{jk}} + b_{1_{jk}} q^{-1} + \cdots + b_{n_{jk}} q^{-n_{jk}}),
\]

\[
A_{j}(q, \theta) = 1 + a_1 q^{-1} + \cdots + a_{n_{b}} q^{-n_{b}},
\]

Note that all modules \( G_{jk}, k \in D_j \) have the same denominator, and that \( B_{jk}(q, \theta) \) is a polynomial of order \( n_k \) and \( A_{j}(q, \theta) \) is a polynomial of order \( n_{b} \). For notational convenience, in the remainder of this paper, all polynomials \( B_{jk}(q, \theta) \) will be assumed to be of the same order, denoted \( n_b \). Let each module \( G_{jk}, k \in D_j \) be expressed as \( \frac{\theta_{jk}^0}{\theta_{jk}^0} \). Then, from (1), \( w_{j} \) can be expressed using transfer functions with a common denominator as follows:

\[
w_{j}(t) = \frac{1}{A_{j}^0(q)} \sum_{k \in N_j} \tilde{B}_{jk}^0(q) w_k(t) + v_j(t)
\]

where

\[
\tilde{A}_{j}^0(q) = \prod_{n \in N_j} A_{jn}^0(q) \text{ and } \tilde{B}_{jk}^0(q) = \prod_{n \in N_j \setminus k} B_{jk}^0(q) A_{jn}^0(q).
\]

From (16) and (4), the prediction error is:

\[
\epsilon_{j}(\theta) = A_{j}(q, \theta) \tilde{w}_{j}(t) - \sum_{k \in D_{j}} B_{jk}(q, \theta) \tilde{w}_{k}(t)
\]

\[
= \tilde{w}_{j}(t) \left[ \tilde{\phi}_{k1}^T(t) \cdots \tilde{\phi}_{kn}^T(t) \tilde{\phi}_{j}^T(t) \right] \theta
\]

\[
= \tilde{w}_{j} - \tilde{\phi}_{j}^T(t) \theta.
\]

where

\[
\tilde{\phi}_{j}^T(t) = [\tilde{w}_{k1}(t) \cdots \tilde{w}_{kn}(t-n_k) \tilde{\phi}_{j1}(t) \cdots \tilde{\phi}_{jn}(t-n)] \text{ and } \theta \text{ is a vector of parameters defined analogously to (8)}.
\]

The result of Proposition 1 can now be extended to the case of dynamic networks where only noisy measurements of the internal variables are available.
Proposition 7 Consider a dynamic network as defined in Section 2.1 that satisfies Assumption 1. Consider the prediction error (18). Choose the set of predictor inputs, \( \{w_1, \ldots, w_k\} \), such that \( \{k_1, \ldots, k_d\} = N_j \) (i.e. \( \mathcal{D}_j = N_j \)). Let \( d = \text{card} (\mathcal{D}_j) \). Choose sets \( \mathcal{I}_j \) and \( X_j \) of instrumental variables according to the methods of (13) - (15) such that \( \mathcal{I}_j \cap \{\mathcal{D}_j \cup \{j\}\} = \emptyset \). The equivalence relation
\[
R_{zz}(\tau) = 0, \quad \tau = 0, \ldots, n_z \quad \iff \quad \{G_{jk}(q, \theta) = G_{jk}^0(q), \forall k \in \mathcal{D}_j \}\tag{19}
\]
holds for any finite \( n_z \geq \lceil (n_a + d n_b)/(\text{length}(z(t))) \rceil \) if the following conditions are satisfied:

(a) If \( v_i \) is present, then there is no path from \( w_j \) to any \( w_k \), \( k \in \mathcal{I}_j \)
(b) The \((d n_b + n_a) \times \text{length}(z)\) matrix
\[
\tilde{R} = \mathbb{E} \left[ \phi(t) \begin{bmatrix} z^T(t) & z^T(t-1) & \cdots & z^T(t-n_z) \end{bmatrix} \right]
\]
is full row rank, where \( \phi(t) \) is defined in (18).
(c) Each sensor noise \( s_t, \ell \in \mathcal{I}_j \) is uncorrelated to all \( s_k, k \in \mathcal{D}_j \).
(d) If \( v_j \) is present, then it is uncorrelated to all \( v_m \) with a path to \( w_j \).
(e) The parameterization is flexible enough, i.e. there exists a \( \theta \) such that \( G_{jk}(q, \theta) = G_{jk}^0(q), \forall k \in \mathcal{D}_j \).

The proof can be found in Appendix B. Most importantly, the presence of sensor noise does not affect the validity implication (19) (as long as Condition (c) holds). Condition (a) puts a restriction on which internal variables are candidate instrumental variables. For example, the candidate instrumental variables cannot be part of any loop that passes through \( w_j \). Note that no similar condition is explicitly stated for the external variables chosen as instrumental variables. This is because, by definition, there is no path from \( v_j \) to any \( r_k \).

As was done when analyzing the BCLIV method in Section 2.3, Condition (b) of Proposition 7 can be further analyzed using the concept of persistence of excitation. Suppose that none of the chosen instrumental variables satisfy all conditions of Lemma 6. This alone does not guarantee that the matrix \( \tilde{R} \) of Condition (b) of Proposition 7 is full rank. For the condition to hold, it additionally must be that the vector \( z \) of chosen instrumental variables is persistently exciting of sufficiently high order. This is formalized in the following lemma, which is the network counterpart to Lemma 3.

\[\text{Lemma 8} \quad \text{Consider the situation of Proposition 7 and} \]
\[
\tilde{R} = \mathbb{E} \left[ \phi(t) \begin{bmatrix} z^T(t) & z^T(t-1) & \cdots & z^T(t-n_z) \end{bmatrix} \right]
\]
where \( z \) is the vector of instrumental variables, and \( \phi(t) \) is defined in (11). Let \( n_\theta \) denote the size of the vector \( \phi(t) \). The matrix \( \tilde{R} \) generically has rank \( n_\theta \) (i.e. full row rank) if the following conditions hold:

(a) \( n_z \cdot \text{length}(z(t)) \geq n_\theta \).
(b) \( z \) is persistently exciting of order \([n_\theta/\text{length}(z)]\).
(c) The parameter vector \( \theta \) such that \( G_{jk}(q, \theta) = G_{jk}^0, \forall k \in \mathcal{D}_j \) is unique.
(d) No instrumental variable satisfies all the conditions of Lemma 6 for all \( w_k, k \in \mathcal{D}_j \cup \{j\} \).

The proof follows the same reasoning as that of Lemma 3. The main point of Lemma 8 is that as long as the instruments are correlated to the predictor inputs and \( w_j \), and are persistently exciting of sufficiently high order, then Condition (b) of Proposition 7 (generically) holds. There is no explicit restriction on the number of instrumental variables, as long as the chosen \( z \) is persistently exciting of sufficiently high order. However, if only one internal variable is selected as the instrumental variable, then by Condition (b) of Lemma 8 \( z \) must be persistently exciting of order \( n_\theta \). Whereas, if two internal variables are selected as instrumental variables then by the same condition \( z \) need only be persistently exciting of order \( n_\theta/2 \). Thus Condition (b) may implicitly place a restriction on the required number of instrumental variables.

Example 1 Consider the data generating system shown in Fig. 1a. Suppose that the objective is to obtain a consistent estimate of \( G_{32}^0 \). Thus, \( \{j\} = \{3\} \), and \( N_3 = \{2\} \).

---

\[\text{Example 2} \quad \text{Consider the data generating system shown in Fig. 1b. Suppose that the objective is to obtain a consistent estimate of \( G_{32}^0 \). Thus, \( \{j\} = \{3\} \), and \( N_3 = \{2\} \).} \]
3. Choose an ARX model structure and construct the model.

2. Choose the sets \( \{ \mathcal{D}_j \cup \{ j \} \cap \mathcal{I}_j = \emptyset \} \), i.e. \( w_j \) must be chosen as the instrumental variable. Since there is no path from \( w_j \) to \( w_1 \), Condition (a) of Proposition 7 holds. Moreover, Condition (b) of Proposition 7 generically holds because the instrumental variable is persistently exciting of sufficiently high order (since \( v_1 \) is white noise) and because there is a path from \( w_1 \) to both \( w_2 \) and \( w_3 \) (i.e. there is a path from the instrumental variable to the predictor inputs and \( w_j \)). If the remaining conditions of Proposition 7 hold, then the implication (19) holds. \( \Box \)

Example 2 Consider the data generating system shown in Fig. 1b. Suppose that the objective is to obtain a consistent estimate of \( G_{0j}^3 \). In this case it is not possible to satisfy Condition (a) of Proposition 7. \( \Box \)

The following algorithm shows how the implication of Proposition 7 can be exploited to obtain an estimate of a module embedded in a dynamic network.

Algorithm 2 Objective: obtain an estimate of \( G_{0j}^3 \).

1. Choose the set of predictor inputs \( \{ w_k, k \in \mathcal{N}_j \} \).
   (i.e. \( \mathcal{D}_j = \mathcal{N}_j \)).
2. Choose the sets \( \mathcal{I}_j \) and \( \mathcal{X}_j \) of instrumental variables.
   Construct \( z \), the vector of instrumental variables.
3. Choose an ARX model structure and construct the prediction error (18).
4. Find a solution, \( \theta_N \) to the set of equations
   \[
   \frac{1}{N} \sum_{t=0}^{N-1} \varepsilon_j(t, \theta) z^T(t - \tau) = 0, \text{ for } \tau = 0, \ldots, n_z, \quad (20)
   \]
   where \( n_z \cdot \text{length}(z(t)) \geq n_a + dm_b \).

This algorithm is similar to that of the BCLIV method (Algorithm 1). Only Steps 1 and 2 are more involved due to the increased complexity of a network vs. a closed loop. Let \( \hat{R}_{ez}(\tau) \) denote the function in (20). Under weak general convergence conditions of the Prediction-Error Identification methods (Ljung, 1999) it follows that

\[
E[\hat{R}_{ez}(\tau)] \to R_{ez}(\tau) \text{ as } N \to \infty \quad (21)
\]

and that the solution to (20), denoted \( \hat{\theta}_N \) tends to \( \theta_0 \) as \( N \to \infty \). Thus, the estimates of \( G_{0j}^3, k \in \mathcal{N}_j \) obtained by Algorithm 2 are consistent if the conditions presented in Proposition 7 are satisfied. In Step 4 of Algorithm 2 \( \hat{\theta}_N \) can be obtained by linear regression. This follows from (20) which is affine in \( \theta \).

In the following section this method is generalized so that it can be used in the situation where there is a path from \( w_j \) to one or more instrumental variables (as was the case in Example 2).

4 Generalized Instrumental Variable Approach

In the previous section the set of candidate instrumental variables is restricted by Condition (a) of Proposition 7, i.e. it is only allowed to choose instrumental variables for which there is no path from \( w_j \) to the instrumental variable. As illustrated in Example 2 this can be a restrictive condition. In this section a method is proposed for which all external variables and all internal variables \( w_\ell, \ell \notin \mathcal{D}_j \cup \{ j \} \) are candidate instrumental variables.

The key difference in this method is that a Box-Jenkins model structure is used instead of an ARX model structure. In this sense the method presented here is a generalization of the classical IV methods. This change is in line with closed-loop identification reasoning where it is well known that for direct methods, consistent estimates are possible if the process noise is correctly modeled (Forssell & Ljung, 1999). The price for the increased applicability is that the estimates of \( G_{0j}^3, k \in \mathcal{N}_j \) can no longer be obtained by solving a linear regression problem.

The main reason that a path from \( w_j \) to the instrumental variable \( w_j \) causes a problem is because then the projections of the predictor inputs onto the instrumental variable(s) are correlated to the output noise. This is equivalent to the closed-loop identification problem where the plant input is correlated to the output noise. From the closed-loop identification literature, there are several methods to deal with this correlation that is induced by feedback (Forssell & Ljung, 1999; Van den Hof et al., 2013). One method, called the Direct Method, deals with the problem by exactly modeling the noise. In the following text it is shown that this idea can be extended to the IV framework, so that all (measured) internal variables \( w_\ell, \ell \in \{ 1, \ldots, L \} \setminus \{ \mathcal{D}_j \cup \{ j \} \} \) are candidate instrumental variables. Note that the idea is to exactly model the process noise term \( v_j \), and not the sensor noise (or a sum of the two). The sensor noise is dealt with using the instrumental variable mechanism.

To exactly model the noise, a Box-Jenkins model structure is required. This amounts to the parameterization:

\[
G_{jk}(q, \theta) = \frac{B_{jk}(q, \theta)}{F_{jk}(q, \theta)} \quad k \in \mathcal{D}_j \quad (22)
\]

where \( F_{jk}(\theta), B_{jk}(\theta), C_j(\theta), D_j(\theta) \) are polynomials in \( q \) of orders \( n_{df}^j, n_{bf}^j, n_c \) and \( n_d \) respectively. For notational convenience all transfer functions \( G_{jk}(q, \theta) \) will be assumed to be of the same orders, denoted \( n_f \) and \( n_b \).

In the following proposition it is shown that by changing the model structure, the fundamental mechanism on which the IV methods are based, holds for the set of candidate instrumental variables \( w_\ell, \ell \in \{ 1, \ldots, L \} \setminus \{ \mathcal{D}_j \cup \{ j \} \} \).
Proposition 9 Consider a dynamic network as defined in Section 2.1 that satisfies Assumption 1. Consider the prediction error (4) and model structure (22). Choose the set of predictor inputs such that \( \mathcal{D}_j = \mathcal{N}_j \). Choose the sets \( \mathcal{I}_j \) and \( \mathcal{X}_j \) of instrumental variables according to the methods of (13) - (15) such that \( \mathcal{I}_j \cap \mathcal{D}_j \cup \{j\} = \emptyset \). Let \( z \) denote the vector of instrumental variables. The equivalence relation
\[
\{ R_{xz}(\tau) = 0, \forall \tau \geq 0 \} \iff \left\{ G_{jk}(q, \theta) = G_{jk}^0(q) \forall k \in \mathcal{D}_j, \ H_j(q, \theta) = H_j^0(q) \right\} \quad (23)
\]
holds if the following conditions are satisfied:
(a) Every instrumental variable \( \tilde{w}_i, \ell \in \mathcal{I}_j \) is a function of only delayed versions of \( w_j \).
(b) Let \( d = \text{card}(\mathcal{D}_j) \). Let \( n_i = \lfloor (d + 1)/\text{length}(z(t)) \rfloor \). Let \( n = \max(n_i + n_f, n_c + n_d) \).
\[
w(t) = [w_{k_1}(t) \cdots w_{k_d}(t) w_j(t)]^T, \{k_1, \ldots, k_d\} = \mathcal{D}_j \\
z'(t) = [z(t) z(t - n_y - 1) \cdots z(t - n_z n_y - 1)]^T
\]
The cross power spectral density \( \Phi_{wz'} \) is full row rank for at least \( n_j \) distinct frequencies \( \omega \in (-\pi, \pi) \).
(c) Every sensor noise variable \( s_k, k \in \mathcal{D}_j \cup \{j\} \) is uncorrelated to every \( w_i, \ell \in \mathcal{I}_j \).
(d) The process noise variable \( v_j \) is uncorrelated to all \( v_k \) with a path to \( w_j \).
(e) The parameterization is chosen flexible enough, i.e. there exists a parameter \( \theta \) such that \( G_{jk}(q, \theta) = G_{jk}^0(q), \forall k \in \mathcal{D}_j \), and \( H_j(q, \theta) = H_j^0(q) \).

The proof can be found in Appendix C. Condition (a) of Proposition 9 can be satisfied in two ways. First, if there is a delay in the path from \( w_i \) to the instrumental variable \( \tilde{w}_i \), then \( w_i \) is only a function of delayed versions of \( w_j \). Secondly, instead of using \( \tilde{w}_i(t) \) as an instrumental variable, it is also possible to use a delayed version of \( w_i(t - 1) \), as an instrumental variable. In this way Condition (a) can be satisfied.

By Condition (e) the process noise must be exactly modelled. This condition is a signature of the Direct closed-loop method Forsell & Ljung (1999); Van den Hof et al. (2013). This is why we can think of the mechanism proposed in Proposition 9 as a hybrid between the Direct closed-loop method and an instrumental variable method. Recall that in Proposition 7 exact noise modeling was not required.

In Proposition 9 Condition (b) is a condition on the data. For the sake of argument, suppose that \( z' \) and \( w \) are the same length. Another interpretation of persistence of excitation. A necessary condition for Condition (b) to hold is that no instrumental variable satisfies all the conditions of Lemma 6 for all \( w_k, k \in \mathcal{D}_j \cup \{j\} \). Consequently, the vector \( w \) is a function of \( z' \) and thus
\[
\Phi_{wz'}(\omega) = K(e^{j\omega})\Phi_{z'}(\omega).
\]
Suppose that \( \text{det}(K) \) has no zeros on the unit circle. Then, if \( \Phi_{z'} \) is full rank for at least \( n \) distinct frequencies, \( \Phi_{wz'} \) will be as well. By Proposition 5 if \( \Phi_{z'} \) is full rank for at least \( n \) distinct frequencies, then \( z' \) is persistently exciting of order \( n \). Thus, we can link Condition (b) of Proposition 9 to the idea of a persistently exciting vector of instrumental variables.

The following examples illustrate the result.

Example 3 Consider again the situation of Example 2. Suppose that there is a delay in \( G_{13}^0 \). Choose, \( \{j\} = \{3\}, \mathcal{N}_2 = \{2\} \). Choose \( w_1 \) as the instrumental variable, i.e. \( z(t) = \tilde{w}_1(t), \mathcal{I}_j = \{1\}, \) and \( \mathcal{N}_j = \emptyset \). Condition (a) is satisfied due to the delay in \( G_{13}^0 \). By Lemma 6, since there is a path from \( w_1 \) to both \( w_2 \) and \( w_3 \), the necessary conditions for Condition (b) to hold are satisfied. If the remaining conditions of Proposition 9 are satisfied, then the implication (23) holds.

The important point, is that for this data generating system, the implication (19) of Proposition 7 did not hold because it was not possible to satisfy Condition (a) of Proposition 7.

\[ \begin{array}{c}
G_{21}^0 \\
G_{17}^0 \\
G_{17}^0 \\
G_{17}^0 \\
G_{17}^0 \\
G_{17}^0 \\
\end{array} \]
\[ \begin{array}{c}
G_{12}^{-1} \\
G_{12}^{-1} \\
G_{12}^{-1} \\
G_{12}^{-1} \\
G_{12}^{-1} \\
G_{12}^{-1} \\
\end{array} \]
\[ \begin{array}{c}
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
\end{array} \]
\[ \begin{array}{c}
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
\end{array} \]
\[ \begin{array}{c}
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
\end{array} \]
\[ \begin{array}{c}
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
G_{13}^{-1} \\
\end{array} \]

Fig. 2. Example of a dynamic network. The sensor noise is not shown in the figure, but it is still assumed that in the available data set, each internal variable is measured with sensor noise. The labels of the \( w_i \)’s have been placed inside the summations indicating that the output of the sum is \( w_i \).

Example 4 Consider the network shown in Fig. 2. Suppose that the objective is to obtain a consistent estimate of \( G_{21}^0 \). Thus, \( \{j\} = \{2\} \), and \( \mathcal{N}_2 = \{1,4,6\} \). Now choose the instrumental variables. One option is to choose 4 distinct internal variables. A possible choice can simply be truncated so that \( z' \) and \( w \) are the same length since \( z \) is a user constructed vector
for the set of instrumental variables is \( \tilde{w}_3(t), \tilde{w}_8(t), \tilde{w}_7(t) \) and \( \tilde{w}_8(t) \) (i.e. \( D_2 = \{3, 5, 7, 8\} \)). In this case if \( z(t) = [\tilde{w}_3(t) \, \tilde{w}_8(t) \, \tilde{w}_7(t) \, \tilde{w}_8(t)]^T \) is persistently exciting of sufficiently high order, and the remaining conditions of Proposition 9 hold, then the implication (23) holds.

Another option for choosing the instrumental variables is to only use \( \tilde{w}_7 \). In this case if \( z(t) = [\tilde{w}_7(t) \, \tilde{w}_7(t - n_0 - 1) \, \tilde{w}_7(t - 2n_0 - 1) \, \tilde{w}_7(t - 3n_0 - 1)] \) is persistently exciting of sufficiently high order, and the remaining conditions of Proposition 9 hold, then the implication (23) holds. Other options for choosing \( z \) are also possible depending on the persistence of excitation of \( z \). □

In the following algorithm the implication of Proposition 9 is exploited to construct a method to obtain an estimate of a module embedded in a dynamic network.

**Algorithm 3** Objective: obtain an estimate of \( G_{0j}^0 \).

1. Choose the set of predictor inputs as \( D_j = \mathcal{N}_j \).
2. Choose the set \( \mathcal{I}_j \) and construct the vector in instrumental variables, \( z \).
3. Choose a Box-Jenkins model structure, (22), and construct the prediction error (4).
4. Find a solution to the set of equations
   \[
   \frac{1}{N} \sum_{t=0}^{N-1} \varepsilon(t, \theta) z(t - \tau) = 0, \quad \text{for } \tau = 1, \ldots, n. \tag{24}
   \]

By Proposition 9 and the reasoning of (21) it follows that the estimate obtained using Algorithm 3 is consistent, as long as all the conditions of the proposition are satisfied.

In the following section the choice \( D_j = \mathcal{N}_j \) is relaxed.

5 Predictor Input Selection

In this paper thus far, the required set of predictor inputs has not been a user choice. In the reasoning thus far, in order to identify a particular module \( G_{0j}^0 \), the internal variables with direct connections to \( w_j \), i.e. \( \tilde{w}_k, k \in \mathcal{N}_j \) must be used as the prediction error (4).

This is an overly restrictive requirement. There is a strong motivation to seek less restrictive conditions. For instance it may be that several internal variables are difficult or expensive to measure and it would be preferred to avoid the necessity of measuring these variables.

The problem of predictor input selection is thoroughly discussed in Dankers et al. (2014b) where conditions are derived that the set of predictor inputs must satisfy in order to ensure that it is possible to consistently identify a module of interest, \( G_{ji}^0 \). In this section those ideas are extended to the methods of Sections 3 and 4.

The conditions can be used to determine whether it is possible to consistently identify a particular transfer function given a set of measured internal variables. Conversely, the conditions can also be used to design a sensor placement scheme to ensure that a particular transfer function can be consistently estimated. The sensor placement scheme could be designed to minimize the number of sensors, or it could be designed to avoid using a particular variable that is difficult to measure. The cost of the increased flexibility in the choice of predictor inputs is that instead of consistently estimating \( G_{0j}^0 \), \( k \in \mathcal{N}_j \), only the module of interest \( G_{0j}^0 \) is consistently estimated.

Before continuing, consider the following definition of a confounding variable. A confounding variable induces a correlation between two noise variables.

**Definition 10** Consider a particular output variable \( w_j \) and a set \( \mathcal{D}_j \) of predictor inputs. In this modeling setup, a variable \( v_m \) is a confounding variable if

(a) there is a path from \( v_m \) to \( w_j \) that does not pass through any \( w_k, k \in \mathcal{D}_j \),
(b) there is a path from \( v_m \) to one or more \( w_k, k \in \mathcal{D}_j \) that does not pass through any \( w_n, n \in \mathcal{D}_j \).

For an example of a confounding variable see Example 3 in Dankers et al. (2013).

Most of the details of Dankers et al. (2014b) are omitted here. Only a key result of Dankers et al. (2014b) is presented. In Dankers et al. (2014b) it is shown that in order to consistently identify the module \( G_{0j}^0 \) the set of predictor inputs must satisfy the following conditions.

**Property 1** Consider the internal variables \( w_i, w_j \) and the set of indices of predictor inputs, \( \mathcal{D}_j \). Let \( \mathcal{D}_j \) satisfy the following conditions:

(a) \( i \in \mathcal{D}_j, j \notin \mathcal{D}_j \),
(b) every loop from \( w_j \) to \( w_i \) passes through a \( w_k, k \in \mathcal{D}_j \),
(c) every path from \( w_i \) to \( w_j \), excluding the path \( G_{0j}^0 \), passes through a \( w_k, k \in \mathcal{D}_j \),
(d) there are no confounding variables in the modeling setup.

5.1 Predictor Input Selection - Extended BCLIV

Consider the following generalization of Proposition 7.

**Proposition 11** Consider a dynamic network as defined in Section 2.1 that satisfies Assumption 1. Consider the MISO ARX model structure (16) and the prediction error (18). Let \( G_{ji}^0 \) denote the module of interest. Choose \( \mathcal{D}_j \) such that it has Property 1. Let \( d = \text{card}(\mathcal{D}_j) \). The implication

\[
\{ R_{zz}(\tau) = 0, \quad \text{for } \tau = 0, \ldots, n \} \implies \{ G_{ji}(q, \theta) = G_{ji}^0(q) \}
\]
holds for any finite $n \geq \lfloor(n_a + d_{na})/\text{length}(z(t))\rfloor$ if the following conditions are satisfied:

(a) Conditions (a) - (c) of Proposition 7 hold.
(b) All process noise variables are uncorrelated to each other.
(c) The parameterization is flexible enough, i.e. there exists a $\theta$ such that $R_{cz}(\tau, \theta) = 0$.

Remark 12 Condition (b) can be somewhat relaxed. See Proposition 7 in Dankers et al. (2014b) for details.

In Algorithm 2 only Step 1 is changed. It this case $D_j$ must be chosen such that it satisfies Condition 1. Thus, by applying Algorithm 2 with a set $D_j$ that has Property 1 a consistent estimate of $G_{ji}^0$ is obtained. In this case, we no longer make a statement about the remaining transfer functions $G_{jk}(q, \theta)$, $k \in D_j \setminus \{i\}$ that need to be estimated. For a interpretation of these remaining transfer functions see Dankers et al. (2014b).

5.2 Predictor Input Selection - Generalized IV

Similarly, the method of Section 4 can be generalized to allow for a more flexible choice of predictor inputs.

Proposition 13 Consider a dynamic network as defined in Section 2.1 that satisfies Assumption 1. Consider the BJ model structure (22) and the prediction error (4). Let $G_{ji}^0$ denote the module of interest. Choose $D_j$ such that it has Property 1. The implication

$$\left\{ R_{cz}(\tau) = 0, \forall \tau \geq 0 \right\} \implies \left\{ G_{ji}(q, \theta) = G_{ji}^0(q) \right\} \quad (25)$$

holds if the following conditions are satisfied:

(a) Conditions (a) - (d) of Proposition 9 hold.
(b) All process noise variables are uncorrelated to each other.
(c) The parameterization is flexible enough, i.e. there exists a $\theta$ such that $R_{cz}(\tau, \theta) = 0$.

The idea of predictor input selection is illustrated in the following example. The example illustrates the additional flexibility that is allowed in choosing the set $D_j$.

Example 5 Consider the network shown in Fig. 3. Suppose that the objective is to obtain an estimate of $G_{32}^0$. First we must choose which internal variables to include as predictor inputs, i.e. we must choose $D_3$ such that it has Property 1. By Condition (a) of Property 1 $w_2$ must be included as a predictor input. Next, we must check all loops from $w_2$ to $w_3$. All such loops pass through $w_2$, which is already chosen as a predictor input, so Condition (b) of Property 1 is satisfied. Next, we check all paths for instance. It can be seen that all paths from $w_2$ to $w_3$ (not including $G_{32}^0$) pass through $w_4$. Thus Condition (c) is satisfied if we include $w_4$ as a predictor input. For $D_3 = \{2, 4\}$ there are no confounding variables, and so it satisfies all the conditions of Property 1. Note that this is not the only choice of $D_3$ that has Property 1.

For this choice of $D_3$, the candidate instrumental variables are $\{\hat{w}_1, \hat{w}_3, \hat{w}_6\}$. For all these candidates there is a path from $w_j$ to the candidate. Thus, Proposition 11 does not apply and we have to defer to Proposition 13. If measurements of all the candidate instrumental variables are available, we could choose to use them all, i.e. $I_j = \{1, 5, 6\}$. Alternatively, if using the smallest number of measurements is desirable, only one of them could be selected as the instrumental variable.

If the remaining conditions of Proposition 13 are satisfied, then the implication (25) holds. □

6 Implementation of Algorithm 3

An attractive feature of the classical IV methods is that the estimates can be obtained by solving a linear regression problem. When making the move to a BJ model structure, as in the method proposed in Section 4, this property is lost. In this section an implementation of the method presented in Section 4 is presented.

We show that standard tools for identifying Box-Jenkins models can be used to obtain an estimate of the solution to (24). Recall from Section 4 that we are interested in finding $\theta$ such that

$$R_{cz}(\tau, \theta) = 0 \text{ for } \tau = 0, \ldots, n_z.$$
This is equivalent to finding $\theta$ such that
\[ \sum_{\tau=0}^{n_2} R_{zz}(\tau, \theta) = 0. \tag{26} \]

Since (26) is nonnegative for all $\theta$, finding $\theta$ such that (26) holds is equivalent to finding $\theta$ such that
\[ \hat{\theta} = \arg \min_{\theta} \sum_{\tau=0}^{n_2} R_{zz}(\tau, \theta). \tag{27} \]

Note that (27) is a standard sum of squared errors objective function. Now, consider the expression for $R_{zz}(\tau)$:
\[
R_{zz}(\tau) = \mathbb{E} \left[ \left( H^{-1}_j(\theta) \left( \sum_{k \in D_j} w_k(t) - \sum_{k \in D_j} G_{jk}(\theta) w_k(t) \right) \right)^2 \right].
\]

The point is that (28) has the same form as the prediction error using a Box-Jenkins model structure (see (4)), where the “output” is $\hat{R}_{zz}(\tau)$ and the predictor “inputs” are $\hat{R}_{wz}(\tau), k \in D_j$. In practice, $R_{zz}(\tau)$ and $R_{wz}(\tau)$ cannot be exactly computed. However, $\hat{R}_{zz}(\tau)$ for instance can be approximated as:
\[ \hat{R}_{wz}(\tau) = \frac{1}{N} \sum_{\tau=0}^{N} w_j(t)z(t-\tau). \]

Thus, we can compute $\hat{R}_{wz}(\tau)$ and $\hat{R}_{wz}(\tau)$ for $\tau = 0, \ldots, n_2$, $k \in D_j$, resulting in a data set. Now standard identification tools (such as the bj function in the MATLAB identification toolbox) can be used to find $\theta$.

**Example 6** Consider the system shown in Fig. 4. The objective is to obtain an estimate of $G_{21}$ using $\hat{w}_1$ and $\hat{w}_2$. Thus, the output is $\hat{w}_2$, and the predictor input is $\hat{w}_1$. This leaves $\hat{w}_3$ as the only choice for instrumental variable. In this case Algorithm 2 does not apply since there is a path from $\hat{w}_2$ to the instrumental variable $\hat{w}_3$. Thus, we use Algorithm 3. All the noise variables $\eta_k$ and $s_k$, $k = 1, 2, 3$ are simulated as sequences of low-pass filtered white noise. 5000 data points are simulated. Results are shown in Fig. 5. The blue lines denote estimates that are obtained by ignoring the presence of sensor noise and applying the Direct Method of Van den Hof et al. (2013). Clearly these estimates are biased. The red lines denote estimates obtained using the implementation of Algorithm 3 presented in this section with $n_2 = 1000$. The estimates appear consistent, as expected.

7 Model Validation

Once a model is obtained, it is possible to express how confident one is that the obtained model is in fact the model that generated the data. Presumably, $\hat{R}_{zz}(\tau, \hat{\theta}_N)$ is small for all $\tau \geq 0$. However, how can one be sure that it is small enough to be considered “very near to zero”? If the variance of $\hat{R}_{zz}(\tau, \hat{\theta}_N)$ is known, then it is possible to say that $\hat{R}_{zz}(\tau, \hat{\theta}_N)$ is zero with probability $p$. Then, by the implications (19) and (23), it follows that it is possible to address the quality of the estimate $G_{jk}(q, \theta)$.

The steps shown in Söderström & Stoica (1990; 1989); Ljung (1999) can be closely followed in order to obtain the variance of $\hat{R}_{zz}(\tau, \hat{\theta}_N)$. The result is that
\[
\sqrt{N} \hat{R}_{zz}(\tau, \hat{\theta}_N) \in \mathbb{N}(0, P)
\]
where $\mathbb{N}(0, P)$ means that as $N \to \infty$ the distribution of $\sqrt{N} \hat{R}_{zz}(\tau, \hat{\theta}_N)$ tends to a normal distribution with zero mean and variance $P$, where (Ljung, 1999):
\[ P = \sum_{\tau=-\infty}^{\infty} R_e(\tau) R_z(\tau). \]

Let $n_\alpha$ denote the $\alpha$ level of the $\mathcal{N}(0,1)$ distribution. Then it is possible to check if (Ljung, 1999)
\[ |\hat{R}_{zz}(\tau, \hat{\theta})| \leq \sqrt{\frac{P}{N} n_\alpha}. \]

If the inequality holds, then the obtained model is the correct model with probability $\alpha$. 

![Fig. 4. Data generating system considered in Example 6](image)

![Fig. 5. Frequency responses related to the system of Fig. 4: $G_{21}$ (dashed), five realizations of estimated frequency responses using the Direct Method (blue) and the generalized IV Method (red).](image)
to

Since the process noise is modelled in the method of Section 4, the variance of the obtained estimate is likely to be inflated expression for $R_{zz}(\tau)$ is a vector of all the measurement noise terms associated with the instrumental variables. Then Lemma 14 is used to prove the result. Using the notation of Lemma 14, $w_y$ and $w_m$ can be expressed in terms of only process noise terms:

$$w_y(t) = \sum_{n=1}^{L} G_{\text{tr}}^0(q)v_n(t)$$

Consequently the cross power spectral density $\Phi_{w_tw_k}$ is

$$\Phi_{w_tw_k}(\omega) = \sum_{n=1}^{L} G_{\text{tr}}^0(e^{j\omega})\Phi_{v_n}(\omega)G_{\text{tr}}^0(e^{-j\omega}) + G_{\text{tr}}^0(e^{j\omega})\Phi_{v_n}(\omega)G_{\text{tr}}^0(e^{-j\omega}) + G_{\text{tr}}^0(e^{j\omega})\Phi_{v_n}(\omega)G_{\text{tr}}^0(e^{-j\omega})$$

Suppose that none of the Conditions of Lemma 6 hold. By Lemma 14 and Condition (a), $G_{\text{tr}}^0$ is zero. Thus the third term of $\Phi_{w_tw_k}(z)$ is zero. Similarly, by Condition (b) the second term is zero. By Condition (c) for each $n \in \{1, \ldots, L\}$ either $G_{\text{tr}}^0$ or $G_{\text{tr}}^0$ is zero. Thus the first term of $\Phi_{w_tw_k}(z)$ is zero. Consequently, if none of the conditions hold, $w_k$ and $w_y$ are uncorrelated.

**A Proof of Lemma 6**

Consider first the following lemma. For a proof see Mason’s Rules (Mason, 1953), or Van den Hof et al. (2013).

**Lemma 14** Consider a dynamic network as defined in Section 2.1 that satisfies Assumption 1. Let $G_{mn}^0$ be the $(m,n)$th entry of $(I - G_0)^{-1}$. If there are no paths from $n$ to $m$ then $G_{mn}^0$ is zero.

Now follows the proof of Lemma 6.

**PROOF.** The proof proceeds by considering $z_\ell = w_\ell$ (the proof for $z_\ell = r_\ell$ is analogous). First both $w_y$ and $w_y$ are expressed in terms of process noise variables. Then Lemma 14 is used to prove the result. Using (18) $R_{zz}(\tau)$ can be expressed as

$$\mathbb{E}[\xi(t)z(t - \tau)] = \mathbb{E}

Both the predictor inputs and the instrumental variable have a component that is due to the sensor noise. However, By Condition (c) both these components can be removed from the expression of $R_{zz}(\tau)$:

$$R_{zz}(\tau) = \mathbb{E}

\phi_{\phi_s}(t) = \left[ s_{d_1}(t) \cdots s_{d_i}(t-n_k) \cdots s_j(t-1) \cdots s_j(t-n_a) \right]$$

and, similarly, $z_s(t)$ is a vector of all the measurement noise terms associated with the instrumental variables.

**B Proof of Proposition 7**

**PROOF.** The proof proceeds by first deriving a simplified expression for $R_{zz}(\tau)$ and then showing that this expression equals 0 for $\tau = 0, \ldots, n$ if and only if $\theta = \theta^0$. Using (18) $R_{zz}(\tau)$ can be expressed as

$$\mathbb{E}[\xi(t)z(t - \tau)] = \mathbb{E}

\phi_{\phi_s}(t) = \left[ s_{d_1}(t) \cdots s_{d_j}(t-n_k) \cdots s_j(t-1) \cdots s_j(t-n_a) \right]$$

and, similarly, $z_s(t)$ is a vector of all the measurement noise terms associated with the instrumental variables.
From (17) \( w_j(t) \) can be expressed as:

\[
w_j(t) = \theta_0^T \phi(t) + A_j^0(q) v_j(t)
\]

where \( \theta_0 = [\theta_{0,1}^T \ldots \theta_{0,n}^T] \) and \( A_j^0 \) is a vector of the coefficients of \( \Phi_{jk} \). Using this expression for \( w_j \) in \( R_{\varepsilon z}(\tau) \):

\[
R_{\varepsilon z}(\tau) = \mathbb{E} \left[ \phi(t) \phi(t) z(t - \tau) \right] = \mathbb{E} \left[ (\Delta \phi(t) + A_j^0(q) v_j(t)) z(t - \tau) \right] \quad \text{(B.1)}
\]

where \( \Delta \phi = \phi(t) - \phi(0) \).

Condition (a) states that there is no path from any predictor input to any variable chosen as an instrument. This implies that each \( w_{\ell}, \ell \in I_j \) is not a function of \( v_j \). This statement can be proved using Lemma 14 as follows. First, using the notation of Lemma 14, express \( w_{\ell} \) in terms of \( v \):

\[
w_{\ell} = \sum_{k=1}^L G_{nk}^0 v_k.
\]

Since there is no path from \( w_j \) to \( w_{\ell} \), by Lemma 14 \( G_{nk}^0 \) is zero. Thus, \( w_{\ell} \) is not a function of \( v_j \). Consequently, by Condition (d) \( w_{\ell} \) and \( v_j \) are uncorrelated. This leads to the following simplification of (B.1):

\[
R_{\varepsilon z}(\tau) = \mathbb{E} \left[ \phi(t) z(t - \tau) \right] \quad \text{(B.2)}
\]

This is the final expression for \( R_{\varepsilon z}(\tau) \).

It follows immediately from (B.2) that if \( \theta = \theta_0 \) then \( R_{\varepsilon z}(\tau) = 0 \) for all \( \tau \geq 0 \).

It remains to be shown that if \( R_{\varepsilon z}(\tau) = 0 \) for all \( \tau = 0, \ldots, n \) for any finite \( n \geq \max(n_u + d_{nk}/\text{length}(z(t))) \) then \( \theta = \theta_0 \). Consider the set of equations:

\[
R_{\varepsilon z}(0) \quad R_{\varepsilon z}(1) \quad \ldots \quad R_{\varepsilon z}(n) = 0
\]

Then, using (B.2), it follows that

\[
\Delta \theta | R_{\varepsilon z}(0) \quad R_{\varepsilon z}(1) \quad \ldots \quad R_{\varepsilon z}(n) = 0. \quad \text{(B.3)}
\]

The matrix \( [R_{\varepsilon z}(0) \quad R_{\varepsilon z}(1) \quad \ldots \quad R_{\varepsilon z}(n)] \) is either square or has more columns than rows. By Condition (b) it is full row rank. Consequently, the only solution to the equation is \( \Delta \theta = 0 \). This proves the result.

\[\square\]

C Proof of Proposition 9

**PROOF.** First simplified expressions for \( \varepsilon_j \) and \( z \) are derived in order to arrive at a simple expression for \( R_{\varepsilon z}(\tau) \). Then it is shown that this expression equals zero for all \( \tau \geq 0 \) iff \( G_{jk}(\theta) = G_{jk}^0 \).

Consider first an expression for the prediction error. Substitute the expressions for \( \tilde{w}_j \) and \( \tilde{w}_k \) into (4):

\[
\varepsilon_j(\theta) = H_j^{-1}(\theta) \left( \sum_{k \in N_j} G_{jk}^0 w_k + v_j + s_j - G_j(\theta)(w_k + s_k) \right)
\]

\[
= H_j^{-1}(\theta) \left( \sum_{k \in N_j} \Delta G_j(\theta) w_k + \Delta H_j(\theta) v_j + e_j \right)
\]

\[
+ H_j^{-1}(\theta) \left( s_j - \sum_{k \in N_j} G_j(\theta) s_k \right) \quad \text{(C.1)}
\]

with \( \Delta G_j(\theta) = G_j^0 - G_j(\theta) \) and \( \Delta H_j(\theta) = H_j^{-1}(\theta) - H_j^{0-1} \).

Now consider the expression for the instrumental vector:

\[
z(t) = \begin{bmatrix} \tilde{w}_{\ell_1}(t) & \ldots & \tilde{w}_{\ell_n}(t) \end{bmatrix}
\]

\[
= \begin{bmatrix} w_{\ell_1}(t) + s_{\ell_1}(t) & \ldots & w_{\ell_n}(t) + s_{\ell_n}(t) \end{bmatrix}. \quad \text{(C.2)}
\]

In the following text, an expression for \( R_{\varepsilon z}(\tau) \) is derived using (C.1) and (C.2) that is valid for all \( \tau \geq 0 \). Subsequently, this expression is used to prove the proposition.

No measurement chosen as an instrumental variable can be a predictor input (\( N_j \cap I_j = \emptyset \) by the statement of the proposition). Thus, no \( s_i \) that appears in the instrumental variable vector \( z \) (C.2), will appear in the expression for \( \varepsilon_j \), (C.1). By Condition (c) each \( s_k, k \in D_j \) is uncorrelated to all \( s_{\ell_1}, \ell \in I_j \), resulting in:

\[
\mathbb{E} \left[ \varepsilon_j(t, \theta) \cdot z(t - \tau) \right] = \mathbb{E} \left[ \left( H_j^{-1}(q, \theta) \sum_{k \in N_j} \Delta G_j(q, \theta) w_k(t) + \Delta H_j(q, \theta) e_j(t) \right) w_{\ell_1}(t - \tau) \cdots w_{\ell_n}(t - \tau) \right] \quad \text{(C.3)}
\]

By Condition (a) each instrumental variable is a function of only delayed versions of \( v_j \) (and thus delayed versions of \( e_j \)), resulting in the following simplification

\[
\mathbb{E} \left[ \varepsilon_j(t, \theta) \cdot z(t - \tau) \right] = \mathbb{E} \left[ \left( H_j^{-1}(q, \theta) \sum_{k \in N_j} \Delta G_j(q, \theta) w_k(t) + \Delta H_j(q, \theta) v_j(t) \right) w_{\ell_1}(t - \tau) \cdots w_{\ell_n}(t - \tau) \right] \quad \text{(C.4)}
\]

which holds for all \( \tau \geq 0 \). Using a vector notation (C.4) can be expressed as:

\[
R_{\varepsilon z}(\tau) = \mathbb{E} \left[ \Delta X(q, \theta)^T \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_n}(t) \\ v_j(t) \end{bmatrix} \cdot \begin{bmatrix} w_{\ell_1}(t - \tau) \\ \vdots \\ w_{\ell_n}(t - \tau) \end{bmatrix} \right]
\]

\[
R_{\varepsilon z}(\tau) = \mathbb{E} \left[ \Delta X(q, \theta)^T \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_n}(t) \end{bmatrix} \cdot \begin{bmatrix} v_j(t) \\ \vdots \\ v_j(t) \end{bmatrix} \right]
\]

\[
R_{\varepsilon z}(\tau) = \mathbb{E} \left[ \Delta X(q, \theta)^T \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_n}(t) \end{bmatrix} \cdot \begin{bmatrix} v_j(t) \\ \vdots \\ v_j(t) \end{bmatrix} \right]
\]

\[
R_{\varepsilon z}(\tau) = \mathbb{E} \left[ \Delta X(q, \theta)^T \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_n}(t) \end{bmatrix} \cdot \begin{bmatrix} v_j(t) \\ \vdots \\ v_j(t) \end{bmatrix} \right]
\]

\[
R_{\varepsilon z}(\tau) = \mathbb{E} \left[ \Delta X(q, \theta)^T \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_n}(t) \end{bmatrix} \cdot \begin{bmatrix} v_j(t) \\ \vdots \\ v_j(t) \end{bmatrix} \right]
\]
where
\[
\Delta X(q, \theta)^T = \left[ \frac{\Delta G_{jk}(q, \theta)}{H_j(q, \theta)} \cdots \frac{\Delta G_{jk}(q, \theta)}{H_j(q, \theta)} \right] \Delta H_j(q, \theta)
\]
and \{k_1, \ldots, k_d\} = D_j. The variable \( v_j \) can be expressed in terms of internal variables as:
\[
v_j = w_j - \sum_{k \in N_j} G^0_{jk}(q) w_k
\]
and so
\[
\begin{pmatrix}
w_{k_1}(t) \\
\vdots \\
w_{k_d}(t) \\
v_j(t)
\end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ -G^0_{jk_1}(q) & \cdots & -G^0_{jk_d}(q) \end{pmatrix} \begin{pmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_d}(t) \\ w_j(t) \end{pmatrix}
\]
(C.5)
Denote the matrix in (C.5) as \( J^0(q) \). Using this notation,
\[
R_{zz}(\tau) = \mathbb{E} \left[ \Delta X(q, \theta)^T J^0(q) w(t) \cdot [w_{k_1}(t-\tau) \cdots w_{k_d}(t-\tau)] \right]
\]
where \( w(t) = [w_{k_1}(t) \cdots w_{k_d}(t) w_j(t)]^T \). Note that (C.6) is valid for all \( \tau \geq 0 \).

Now, consider the ‘if’ statement. It is shown that if \( G_{jk}(q, \theta) = G^0_{jk}, \) for all \( k \in N_j \) and \( H_j(q, \theta) = H^0_j \), then \( R_{zz}(\tau) = 0 \) for all finite \( \tau \geq 0 \). Let \( \theta_0 \) denote this particular parameter vector (such a parameter vector is guaranteed to exist by Condition (e)). Clearly, \( \Delta G_{jk}(\theta_0) = 0 \) and \( \Delta H_j(\theta_0) = 0 \). Thus, from (C.6),
\[
\mathbb{E}[\epsilon_j(t, \theta_0) \cdot z(t-\tau)] = 0, \quad \text{for all } \tau \geq 0.
\]

Consider the expression for the \( d \)th entry of \( \Delta X(q, \theta) \), where \( d \leq \text{card}(D_j) \):
\[
H_j^{-1}(q, \theta) \Delta G_{jk}(q, \theta) \frac{D_j(q, \theta)}{C^j(\theta)} \frac{B_{jk}(q, \theta)}{F_{jk}(q, \theta)} = \frac{D_j(q, \theta) B^0_{jk}(q) F_{jk}(q)}{C^j(\theta) F^0_{jk}(q) F_{jk}(q, \theta)} = \Delta P_{jk}(\theta) K_{jk}(q, \theta)
\]
(C.9)
where
\[
\Delta P_{jk}(q, \theta) = B^0_{jk}(q) F_{jk}(q, \theta) - B_{jk}(q, \theta) F^0_{jk}(q),
\]
\[
K_{jk}(\theta) = \frac{D_j(q, \theta) C^j(\theta) F^0_{jk}(q) F_{jk}(q, \theta)}{C^j(\theta) F^0_{jk}(q) F_{jk}(q, \theta)}.
\]

Note that \( \Delta P_{jk}(q, \theta) \) is a polynomial of order \( n_f + n_b \) (or less). Similarly, the last entry of \( \Delta X(q, \theta) \) can be expressed as:
\[
\Delta H_j(q, \theta) = \frac{D_j(q, \theta)}{C^j(\theta)} \frac{D^0_j(q)}{C^0_j(q)} - \frac{1}{C^j(\theta) C^0_j(q)} (D_j(q, \theta) C^j(\theta) - D^0_j(q) C^j(\theta)) = \Delta P_j(q, \theta) K_j(q, \theta).
\]
(C.10)
where
\[
\Delta P_j(q, \theta) = D_j(q, \theta) C^j(\theta) - D^0_j(q) C^j(\theta),
\]
\[
K_j(q, \theta) = \frac{1}{C^j(\theta) C^0_j(q)}.
\]

Note the \( \Delta P_j(q, \theta) \) is a polynomial of order \( n_c + n_d \) (or less). For notational convenience suppose that \( n_k \leq n_g \).

The next step is to plug the expressions for \( \Delta X(e^{j\omega}) \), (C.9) and (C.10) into (C.8). Let
\[
\Delta P(e^{j\omega}) = [\Delta P_{jk_1}(e^{j\omega}) \cdots \Delta P_{jk_d}(e^{j\omega}) \Delta P_{j}(e^{j\omega})]
\]
where $\{k_1, \ldots, k_d\} = D_j$, then

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta P(e^{j\omega}, \theta) K(e^{j\omega}) J_0^q(e^{j\omega}) \Phi_{wz}(\omega) \Phi_w H_z(\omega)
\cdot J_0^q(e^{j\omega}) K_T(e^{j\omega}, \theta) \Delta P_T(e^{-j\omega}, \theta) d\omega = 0,
$$

where superscript $H$ denotes conjugate transpose, and

$$
K(\theta) = \text{diag}(K_{jd_1}(\theta), \ldots, K_{jd_n}(\theta), K_j(\theta))
$$

where the argument $e^{j\omega}$ has been dropped for notational clarity, $\text{diag}(\cdot)$ denotes a diagonal matrix with the arguments on the diagonal. Since the term in the integral is nonnegative for all theta, the only way that the integral can equal zero is if the term in the integral is zero for all omega, i.e.:

$$
\Delta P(e^{j\omega}, \theta) \Upsilon(\omega) \Delta P_T(e^{-j\omega}, \theta) = 0 \quad (C.11)
$$

for all $\omega \in [-\pi, \pi]$ where

$$
\Upsilon = K(\theta) J_0^q \Phi_{wz} \Phi_w H_z J_0^q K^H(\theta).
$$

By Condition (b) $\Phi_{wz}$ is full rank at $n_q$ distinct frequencies. Note that $J_0^q$ and $K(\theta)$ are full rank for all $\omega$, and $\theta$. The result is that $\Upsilon$ is positive definite for at least $n_q + 1$ frequencies. However, $\Delta P(q)$ is a (vector) polynomial of degree $n_q$ only. This implies that $\Delta P(q) = 0$ (Söderström & Stoica, 1983).

From the definition of $\Delta P(\theta)$ this implies that $G_{jd}(q, \theta) = G_{jd}(q)$, for all $d \in D_j$ and $H_j(q, \theta) = H_j(q)$. □

References


